Relational depth & applications to transformation and diagram semigroups

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Green's $\mathcal{J}$-relation

> In a semigroup S,

$$(x, y) \in \mathcal{J} \iff S^1 x S^1 = S^1 y S^1.$$

> The $\mathcal{J}$-classes of S are of the form

$$J_x = \{x' \in S : (x', x) \in \mathcal{J}\}.$$

> Some semigroups have $\mathcal{J}$-classes that form a chain ($J_x \leq J_y \iff S^1 x S^1 \subseteq S^1 y S^1$).

Semigroups with chain-like $\mathcal{J}$-classes

> Transformation Semigroups

- T_n – all functions from $[n]$ into $[n]$
- I_n – all partial bijections on $[n]$

  e.g. $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & - & 5 & 6 \end{pmatrix} \in I_6$
- PT_n – all partial transformations on $[n]$

  e.g. $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & - & 2 & 5 & 6 \end{pmatrix} \in PT_6$

> For $S \in \{T_n, I_n, PT_n\}$, $\alpha \mathcal{J} \beta \iff \operatorname{rank}\alpha = \operatorname{rank}\beta$.

- The $\mathcal{J}$-classes of S are of the form

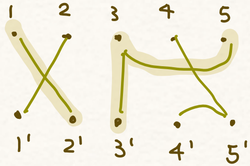
  $$J_i := \{\alpha \in S : \operatorname{rank}\alpha = i\}$$
- The $\mathcal{J}$-classes of S form a chain,

  $$J_\varepsilon < \cdots < J_n \quad (\varepsilon \in \{0,1\}).$$

> Diagram monoids

- Partition monoid P_n – all partitions of $\{1,\dots,n\} \cup \{1',\dots,n'\}$

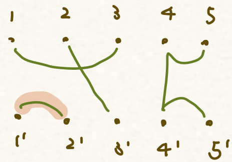
e.g.

$a =$ 

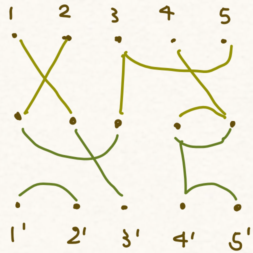
$\in P_5$

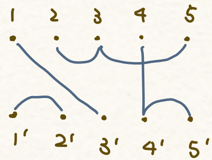
– $\{3,5,3'\}$ is a transversal block.

multiplication:

$b =$ 

$\{1',2'\}$ non-transversal block;
$\{2,3'\}$ transversal block

a
b


$=$ 

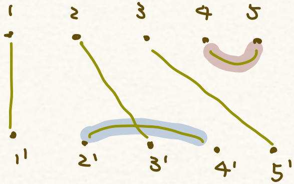
rank = # transversal blocks

> the $\mathcal{J}$-classes of P_n form a chain $J_0 < \dots < J_n$,
where $J_i = \{ a \in P_n : \text{rank } a = i \}$.

- Brauer monoid B_n – all partitions whose blocks have size 2

e.g.

$a =$



$\in B_5$

$\{4,5\}$ – upper non-transversal

$\{2',4'\}$ – lower non-transversal

rank $a = 3$.

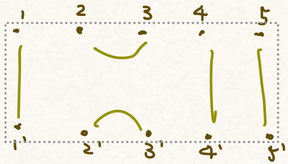
Note: The parity of the number of transversal blocks in elements of B_n is the same as the parity of n.

> For odd n: the J-classes of B_n are $J_1 < J_3 < \cdots < J_n$.

For even n: the J-classes of B_n are $J_0 < J_2 < \cdots < J_n$.

- Temperley-Lieb monoid TL_n

e.g.



$\in TL_5$

$\mathcal{J}$-classes of the ideals of $\mathcal{T}_n$, $\mathcal{I}_n$, $\mathcal{PT}_n$, $\mathcal{P}_n$, $\mathcal{B}_n$

- A subset I of semigroup S is an ideal of S if for all $s \in S$, $i \in I$, si, $is \in I$.

- The ideals of $\mathcal{T}_n$, $\mathcal{I}_n$, $\mathcal{PT}_n$, $\mathcal{P}_n$ are of the form

$$I_m := \{ \alpha \in S : \text{rank } \alpha \leq m \} \quad (\varepsilon \leq m \leq n).$$

The $\mathcal{J}$-classes of I_m are $J_\varepsilon < \cdots < J_m$.

- The ideals of $\mathcal{B}_n$ (n odd) are of the form

$$I_m := \{ \alpha \in \mathcal{B}_n : \text{rank } \alpha = 1, 3, \ldots, m-2, m \}$$

The ideals of $\mathcal{B}_n$ (n even) are of the form

$$I_m := \{ \alpha \in \mathcal{B}_n : \text{rank } \alpha = 0, 2, \ldots, m-2, m \}$$

Semigroup Presentations

> $\langle \underset{\text{alphabet}}{A} \mid \overset{\subseteq A^+ \times A^+}{R} \rangle$ is a presentation for the semigroup S if $S \cong A^+/R^\#$,
>
> where $R^\#$ is the smallest congruence on A^+ that contains R.

Semigroup presentation for S_n

Proposition (Moore, 1897)

The presentation

$$\langle a, b \mid a^2 = b^n = (ba)^{n-1} = (ab^{n-1}ab)^3$$
$$= (ab^{n-j}ab^j)^2 = 1 \quad (2 \le j \le n-2) \rangle$$

defines S_n in terms of generators $a = (1\ 2)$ and $b = (1\ 2\ \dots\ n)$.

Presentation for T_n

Proposition (Aĭzenštat, 1958)

Suppose that $\langle a, b \mid R \rangle$ is a (semigroup) presentation for S_n, where a represents $(1\,2)$, b represents $(1\,2\,\cdots\,n)$. Let t represent the transformation

$$\tau = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 3 & \cdots & n \end{pmatrix} \in T_n.$$

Then the presentation

$$\langle a, b, t \mid R,\ at = b^{n-2}ab^2tb^{n-2}ab^2 = bab^{n-1}abtb^{n-1}abab^{n-1} = (tbab^{n-1})^2 = t,\ (b^{n-1}abt)^2 = tb^{n-1}abt = (tb^{n-1}ab)^2,\ (tbab^{n-2}ab)^2 = (bab^{n-2}ata)^2 \rangle$$

defines T_n.

Aĭzenštat's presentation for T_6

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 4 & 5 & 6 \end{pmatrix}$$

$$b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix}$$

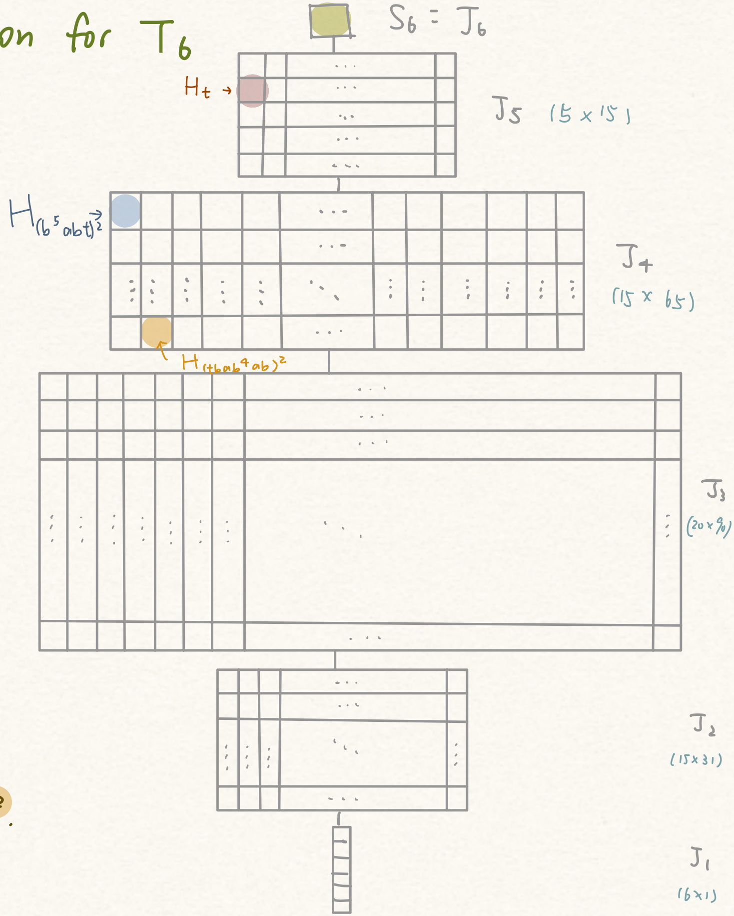
$$t = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 1 & 3 & 4 & 5 & 6 \end{pmatrix}$$

Relations in Aĭzenštat's presentation are:

R, $at = \cdots = t$,

$(b^{n-1}abt)^2 = \cdots = (tb^{n-1}ab)^2$,

$(tbab^{n-2}ab)^2 = (bab^{n-2}ata)^2$.



Presentation for $T_n \setminus S_n$

Proposition (East, 2013)

Define maps α_{ij} $(1 \le i < j \le n)$ by $x\alpha_{ij} = \begin{cases} x & \text{if } 1 \le x < j \\ i & \text{if } x = j \\ x-1 & \text{if } j < x \le n, \end{cases}$ β_i $(1 \le i \le n-1)$ by

$x\beta_i = \begin{cases} x & \text{if } x < i \text{ or } x = n \\ x+1 & \text{if } i \le x < n, \end{cases}$ σ_i $(1 \le i \le n-2)$ by $x\sigma_i = \begin{cases} x & \text{if } x \ne i, i+1, n \\ i+1 & \text{if } x = i \\ i & \text{if } x = i+1 \\ (n-1)\sigma_i & \text{if } x = n \end{cases}$.

Define an alpabet $Y = A \cup B \cup S$, where

$A = \{a_{ij} \mid 1 \le i < j \le n\}$, $B = \{b_i \mid 1 \le i \le n-1\}$, $S = \{s_i \mid 1 \le i \le n-2\}$.

Let Q be the set of relations

$$a_{kl}a_{in} = a_{kl} \quad \text{for all } i,k,l \tag{A1}$$

$$a_{jk}a_{ij} = a_{ik}a_{ij} = a_{ij}a_{i,k-1} \quad \text{if } i < j < k \tag{A2}$$

$$a_{kl}a_{ij} = \begin{cases} a_{ij}a_{k-1,l-1} & \text{if } i < j < k < l & \text{(A3)} \\ a_{ij}a_{k,l-1} & \text{if } i < k < j < l & \text{(A4)} \\ a_{i,j+1}a_{kl} & \text{if } i < k < l \le j < n; & \text{(A5)} \end{cases}$$

$$b_jb_i = b_ib_{j+1} \quad \text{if } 1 \le i \le j \le n-2 \tag{B1}$$

$$b_{n-1}b_i = b_i \quad \text{for all } i; \tag{B2}$$

$$s_ia_{n-1,n} = a_{n-1,n}s_i = s_i \quad \text{for all } i \tag{S1}$$

$$s_i^2 = a_{n-1,n} \quad \text{for all } i \tag{S2}$$

$$s_is_j = s_js_i \quad \text{if } |i-j| > 1 \tag{S3}$$

$$s_is_js_i = s_js_is_j \quad \text{if } |i-j| = 1; \tag{S4}$$

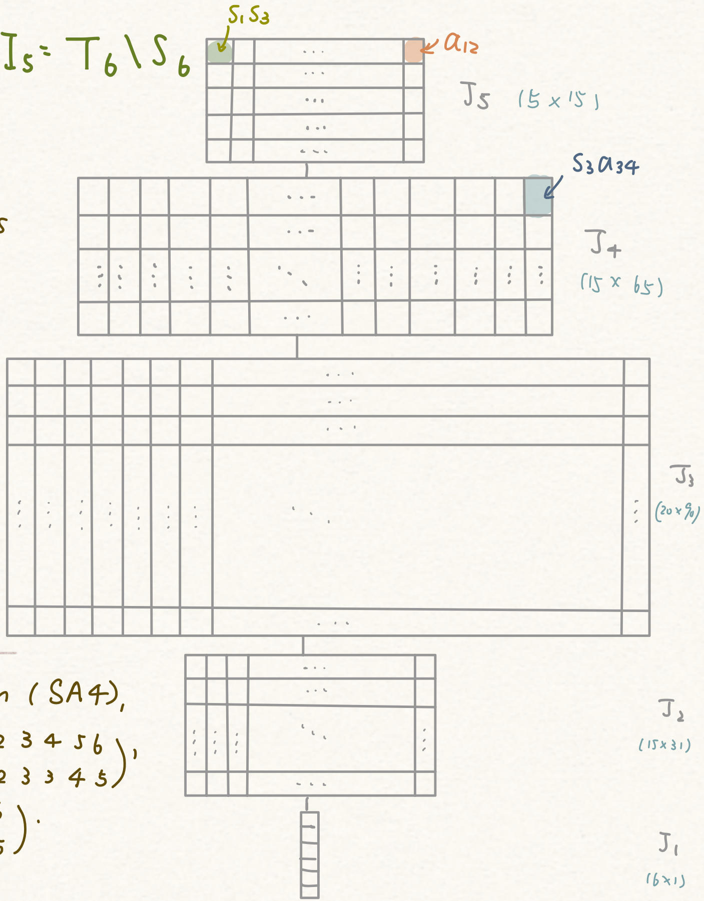
$$s_ra_{ij} = \begin{cases} a_{n-1,n}a_{ij}s_r & \text{if } r \le i-2 \text{ and } j < n & \text{(SA1)} \\ a_{n-1,n}a_{i-1,j}s_r & \text{if } r = i-1 \text{ and } j < n & \text{(SA2)} \\ a_{n-1,n}a_{i+1,j}s_r & \text{if } r = i < j-1 \text{ and } j < n & \text{(SA3)} \\ a_{n-1,n}a_{ij} & \text{if } r = i = j-1 & \text{(SA4)} \\ a_{n-1,n}a_{ij}s_r & \text{if } i < r < j-1 \text{ and } j < n & \text{(SA5)} \\ a_{n-1,n}a_{i,j-1} & \text{if } i < r = j-1 & \text{(SA6)} \\ a_{n-1,n}a_{i,j+1} & \text{if } r = j & \text{(SA7)} \\ a_{n-1,n}a_{ij}s_{r-1} & \text{if } j < r & \text{(SA8)} \\ s_r & \text{if } j = n; & \text{(SA9)} \end{cases}$$

$$b_is_r = \begin{cases} a_{n-1,n}a_{n-2,n-1}s_rb_i & \text{if } r \le i-2 \text{ and } i < n-1 & \text{(BS1)} \\ s_r & \text{if } r \le i-2 \text{ and } i = n-1 & \text{(BS2)} \\ a_{n-1,n}a_{n-2,n-1}b_{i-1} & \text{if } r = i-1 \text{ and } i < n-1 & \text{(BS3)} \\ s_r & \text{if } r = i-1 \text{ and } i = n-1 & \text{(BS4)} \\ a_{n-1,n}a_{n-2,n-1}b_{i+1} & \text{if } r = i & \text{(BS5)} \\ a_{n-1,n}a_{n-2,n-1}s_{i+1} & \text{if } r = i = n-3 & \text{(BS6)} \\ a_{n-1,n}a_{n-2,n-1} & \text{if } r = i = n-2 & \text{(BS7)} \\ a_{n-1,n}a_{n-2,n-1}s_{r-1}b_i & \text{if } i < r; & \text{(BS8)} \end{cases}$$

$$b_ra_{ij} = \begin{cases} a_{n-1,n}a_{i-1,j-1}b_r & \text{if } r < i & \text{(BA1)} \\ s_{j-2}\cdots s_i & \text{if } r = i < j-1 & \text{(BA2)} \\ a_{n-1,n} & \text{if } r = i = j-1 & \text{(BA3)} \\ a_{n-1,n}a_{i,j-1}b_r & \text{if } i < r < j & \text{(BA4)} \\ a_{n-1,n} & \text{if } r = j & \text{(BA5)} \\ a_{n-1,n}a_{ij}b_{r-1} & \text{if } j < r; & \text{(BA6)} \\ a_{n-1,n}a_{ij}b_{r-1} & & \text{(BA7)} \end{cases}$$

The semigroup $T_n \setminus S_n$ has presentation $\langle Y \mid Q \rangle$.

East's presentation for $I_5 = T_6 \setminus S_6$



- $a_{12}a_{16} = a_{12}$ is a relation in (A1), where a_{12} represents

  $$\alpha_{12} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

- $S_3S_1 = S_1S_3$ is a relation in (S3), where S_3 represents

  $$\sigma_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 3 & 5 & 5 \end{pmatrix}.$$ and S_1

  represents $$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 4 & 5 & 5 \end{pmatrix}.$$

- $S_3a_{34} = a_{56}a_{34}$ is a relation in (SA4), where a_{34} represents $\alpha_{34} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 3 & 4 & 5 \end{pmatrix}$, a_{56} represents $\alpha_{56} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 5 \end{pmatrix}$.

Presentation for $\mathcal{I}_n$

Proposition (Meakin, 1993)

Let $\langle a_1, \ldots, a_{n-1} \mid R \rangle$ be a (semigroup) presentation for S_n, where a_i represents the transposition $\alpha_i = (i\ i{+}1)$ for $i = 1, \ldots, n-1$. Let t represent the partial bijection

$$\tau = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-1 & - \end{pmatrix} \in \mathcal{I}_n.$$

Then the presentation

$$\langle a_1, \ldots, a_{n-1}, t \mid R,\ t^2 = t,\ t a_{n-1} t = t a_{n-1} t a_{n-1} = a_{n-1} t a_{n-1} t,$$
$$t a_i = a_i t \ (1 \le i \le n-2) \rangle$$

defines $\mathcal{I}_n$.

Meakin's presentation for I_6

For $1 \le i \le 5$, a_i represents $\alpha_i = (i \;\; i+1)$, t represents

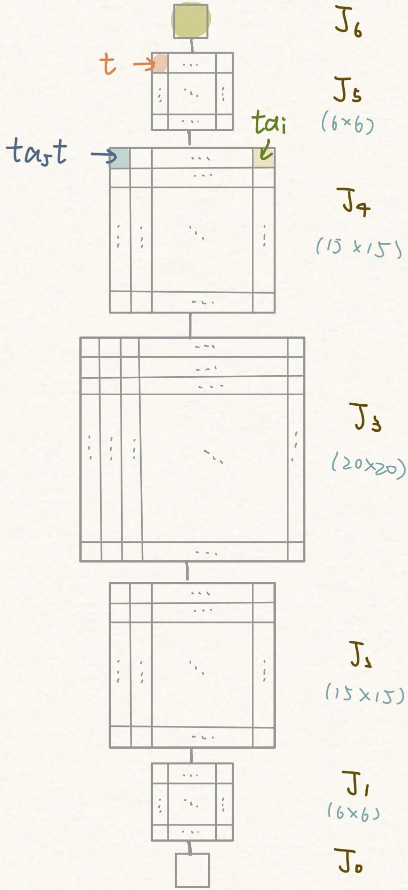
$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & - \end{pmatrix}.$$

The relations in the presentation are:

R, $t^2 = t$,

$ta_5t = \cdots = a_5ta_5t$,

$ta_i = a_it \quad (1 \le i \le n-2)$.



Presentation for $I_n \wr S_n$

Proposition (East, 2006)

For $1 \le i \le n$, define λ_i by $x\lambda_i = \begin{cases} x & \text{if } x < i \\ x+1 & \text{if } i \le x < n \end{cases}$, and ρ_i by

$x\rho_i = \begin{cases} x & \text{if } x < i \\ x-1 & \text{if } i < x \le n. \end{cases}$ For $1 \le j \le n-2$,

define s_i by $x s_i = \begin{cases} x & \text{if } x \neq i, i+1, n \\ i+1 & \text{if } k = i \\ i & \text{if } k = i+1 \end{cases}$.

Define and alphabet $L \cup S \cup R$ where the elements in L, S, R are in one-one correspondence with λ_i's, s_i's, ρ_i's respectively. Then $I_n \wr S_n$ has presentation

$$\langle L \cup S \cup R \mid (L1\text{-}L2), (R1\text{-}R2), (RL1\text{-}RL3), (S1\text{-}S4), (SL1\text{-}SL4), (RS1\text{-}RS4) \rangle.$$

$\lambda_i\lambda_j = \lambda_{j+1}\lambda_i$	if $1 \le i \le j \le n-1$	(L1)
$\lambda_i\lambda_n = \lambda_i$	if $1 \le i \le n$	(L2)
$\rho_j\rho_i = \rho_i\rho_{j+1}$	if $1 \le i \le j \le n-1$	(R1)
$\rho_n\rho_i = \rho_i$	if $1 \le i \le n$.	(R2)

$$\rho_i\lambda_j = \begin{cases} \lambda_n\lambda_{j-1}\rho_i & \text{if } 1 \le i < j \le n \\ \lambda_n = \rho_n & \text{if } 1 \le i = j \le n \\ \lambda_n\lambda_j\rho_{i-1} & \text{if } 1 \le j < i \le n. \end{cases} \qquad \text{(RL1–RL3)}$$

$s_i\lambda_n = \lambda_n s_i = s_i$	for all i	(S1)
$s_i^2 = \lambda_n$	for all i	(S2)
$s_i s_j = s_j s_i$	if $\lvert i-j \rvert > 1$	(S3)
$s_i s_j s_i = s_j s_i s_j$	if $\lvert i-j \rvert = 1$.	(S4)

$$s_i\lambda_j = \begin{cases} \lambda_n\lambda_j s_i & \text{if } 1 \le i < j-1 \le n-2 \\ \lambda_n\lambda_{j-1} & \text{if } 1 \le i = j-1 \le n-2 \\ \lambda_n\lambda_{j+1} & \text{if } 1 \le i = j \le n-2 \\ \lambda_n\lambda_j s_{i-1} & \text{if } 1 \le j < i \le n-2 \end{cases} \qquad \text{(SL1–SL4)}$$

$$\rho_j s_i = \begin{cases} s_i\rho_j\rho_n & \text{if } 1 \le i < j-1 \le n-2 \\ \rho_{j-1}\rho_n & \text{if } 1 \le i = j-1 \le n-2 \\ \rho_{j+1}\rho_n & \text{if } 1 \le i = j \le n-2 \\ s_{i-1}\rho_j\rho_n & \text{if } 1 \le j < i \le n-2. \end{cases} \qquad \text{(RS1–RS4)}$$

East's presentation for $I_5 = I_6 \setminus S_6$

- $s_4\lambda_4 = \lambda_6\lambda_5$ is a relation in (SL3), where

$$s_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 5 & 4 & - \end{pmatrix},$$

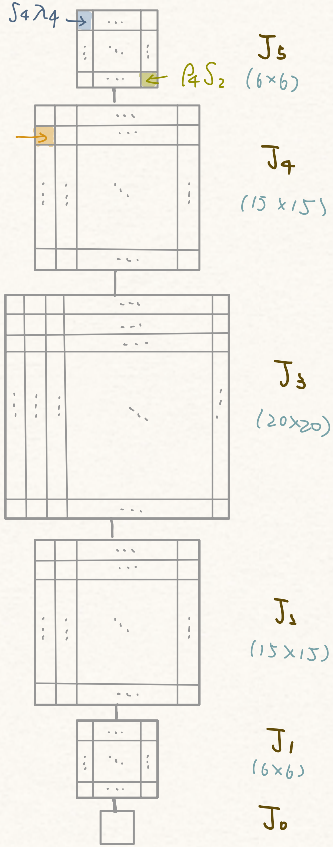
$$\lambda_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 5 & 6 & - \end{pmatrix}.$$

- $\rho_4 s_2 = s_3\rho_4\rho_n$ is a relation in (RS1), where

$$\rho_4 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & - & 4 & 5 \end{pmatrix},$$

$$s_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 4 & 5 & - \end{pmatrix}.$$

- $\rho_4^2 = \rho_4\rho_5$ is a relation in (R1).



Some other presentations for the transformation semigroups and the singular part

- O. Ganyushkin, V. Mazorchuk, Classical Finite Transformation Semigroups (2008) (presentation for $\mathcal{PT}_n$)
- J. East, Defining relations for idempotent generators in finite partial transformation semigroups (2013) ($\mathcal{PT}_n \setminus \mathcal{S}_n$)
- J. D. Mitchell, M. T. Whyte, Short presentations for transformation monoids (2024) (short presentations for $\mathcal{I}_n$, $\mathcal{T}_n$, $\mathcal{PT}_n$)
- J. East, A symmetrical presentation for the singular part of the symmetric inverse monoid (2015) ($\mathcal{I}_n \setminus \mathcal{S}_n$)

Question: How about a general ideal I_k of $\mathcal{T}_n$, $\mathcal{I}_n$, $\mathcal{PT}_n$?

Relational depth

> Consider a finite semigroup S whose $\mathcal{J}$-classes form a chain. let $\mathcal{P} = \langle A \mid R \rangle$ be a presentation for S. let $w \in A^+$, and $(u=v) \in R$.

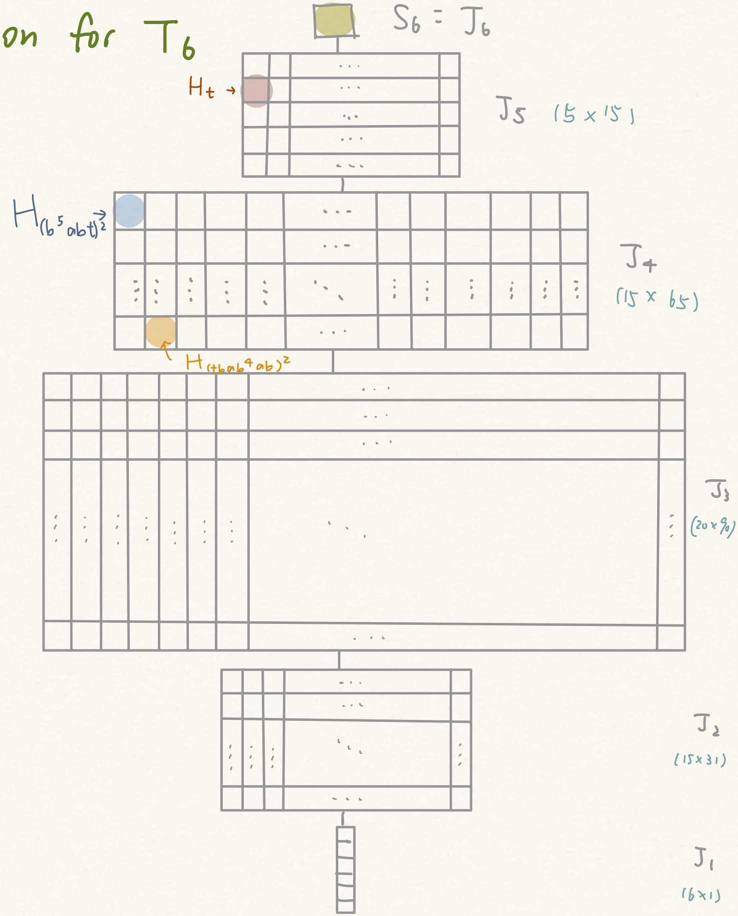
- depth (J) is the minimal length d of a maximal chain of $\mathcal{J}$-classes $J = J_1 < J_2 < \cdots < J_d$ in S.
- depth $(s) :=$ depth (J_s)
- depth (w) is the depth of the element it represents in S
- depth $(u=v) :=$ depth(u) $\quad$ $($ = depth$(v))$
- depth $(\mathcal{P})$ is the maximal depth of a defining relation in $\mathcal{P}$

> The relational depth of S is

$$\text{depth}(S) := \min\{\text{depth}(\mathcal{P}) : \mathcal{P} \text{ is a presentation for } S\}.$$

Arzenštat's presentation for T_6

Relations in Arzenštat's presentation are in the top 3 J-classes, so has depth 3, and depth $(T_n) \leq 3$.



Relational depth of transformation semigroups & ideals

Theorem

Let $n \geq 3$, let $S \in \{T_n, I_n, PT_n\}$. Define $\varepsilon = \begin{cases} 0 & (S = I_n, PT_n) \\ 1 & (S = T_n) \end{cases}$.

For $m \in [\varepsilon, n]$, let I_m be the ideal of all (partial) maps in S of rank at most m. Then

$$\operatorname{depth}(I_m) = \begin{cases} 3 & (m = n) \\ n - m + 1 & (m < n \text{ and } m > \frac{n+\varepsilon}{2}) \\ m - \varepsilon + 1 & (m \leq \frac{n+\varepsilon}{2}) \end{cases}$$

- In other words, when $m \leq \frac{n+1}{2}$, any presentation for an ideal I_m of T_n contains defining relations of constant maps.
- If $m > \frac{n+1}{2}$, a presentation for I_m contains defining relations of transformations of rank $2m - n$.

Example: Relational depth of T_9 and its ideals

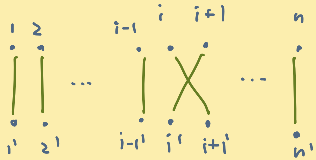
	relational depth	the maximal minimum rank of the defining relations
I_1	1	1
I_2	2	1
I_3	3	1
I_4	4	1
I_5	5	1
I_6	4	3
I_7	3	5
I_8	2	7
T_9	3	7

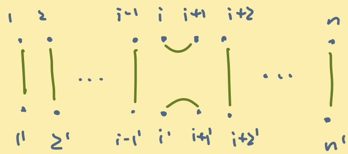
What about diagram semigroups?

Presentation for B_n

Proposition (Kudryavtseva, Mazorchuk, 2006)

For $1 \le i \le n-1$, let s_i denote the elementary transposition $(i, i+1) \in S_n$, and let π_i denote the element $\{i, i+1\} \cup \{i', (i+1)'\} \cup \bigcup_{j \ne i, i+1} \{j, j'\}$.

$s_i = (i, i+1) =$  $\in J_n$

$\pi_i = \{i, i+1\} \cup \{i', (i+1)'\} \cup \bigcup_{j \ne i, i+1} \{j, j'\} =$  $\in J_{n-2}$

Presentation for B_n

Proposition (Kudryavtseva, Mazorchuk, 2006)

For $1 \leq i \leq n-1$, let s_i denote the elementary transposition $(i, i+1) \in S_n$, and let π_i denote the element $\{i, i+1\} \cup \{i', (i+1)'\} \cup \bigcup_{j \neq i, i+1} \{j, j'\}$.

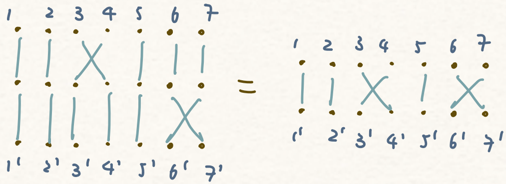
Then the presentation

$$\langle e, s_i, \pi_i \ (i = 1, \dots, n-1) \mid s_i^2 = e;\ s_i s_j = s_j s_i \ (|i-j|>1);\ s_i s_j s_i = s_j s_i s_j \ (|i-j|=1);\ \pi_i^2 = \pi_i;\ \pi_i \pi_j = \pi_j \pi_i \ (|i-j|>1);\ \pi_i \pi_j \pi_i = \pi_i \ (|i-j|=1);\ \pi_i s_i = s_i \pi_i = \pi_i,\ \pi_i s_j = s_j \pi_i \ (|i-j|>1);\ s_i \pi_j \pi_i = s_j \pi_i,\ \pi_i \pi_j s_i = \pi_i s_j \ (|i-j|=1) \rangle$$

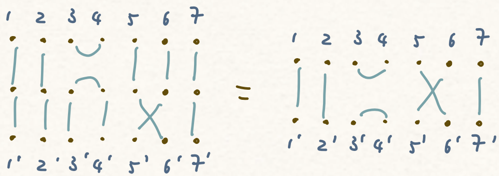
defines B_n.

Presentation for B_7

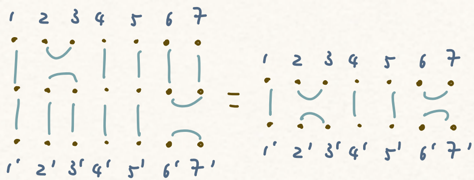
- $S_3 S_6 = S_6 S_3 \in J_7$

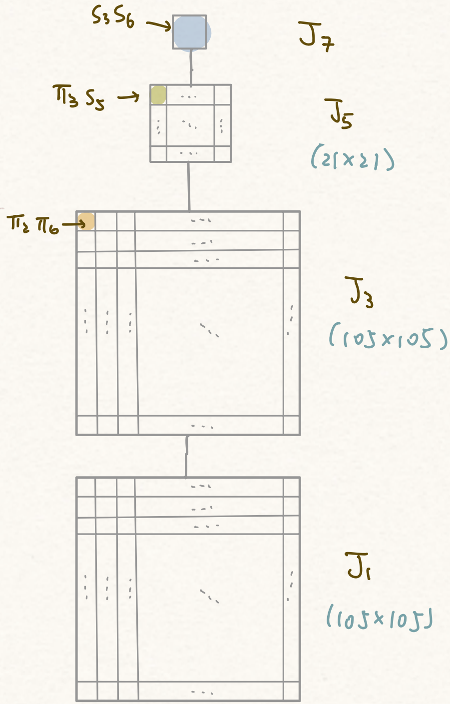


- $\pi_3 S_5 = S_5 \pi_3 \in J_5$



- $\pi_2 \pi_6 = \pi_6 \pi_2 \in J_3$



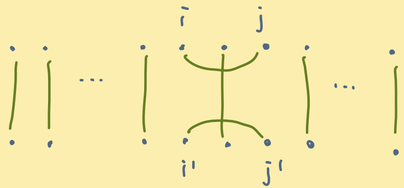


Presentation for $B_n \setminus S_n$

Proposition (Maltcev, Mazorchuk, 2006)

For $i, j \in \{1, \dots, n\}$, and $i \neq j$, define $\sigma_{i,j} = \{\{i,j\}, \{i',j'\}, \{k,k'\}_{k \neq i,j}\}$.

$\sigma_{i,j} =$



$\in J_{n-2}$

Presentation for $B_n \setminus S_n$

Proposition (Maltcev, Mazorchuk, 2006)

For $i, j \in \{1, \dots, n\}$, and $i \neq j$, define $\sigma_{i,j} = \{\{i,j\}, \{i', j'\}, \{k, k'\}_{k \neq i,j}\}$.

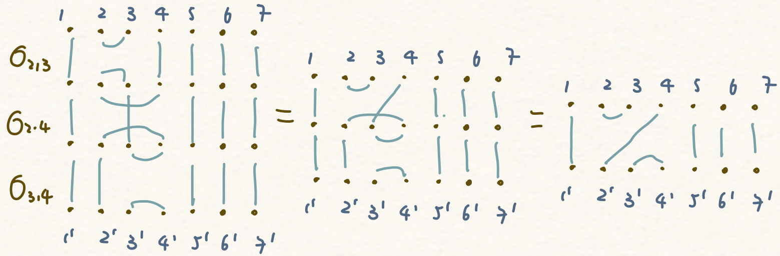
Then the presentation

$$\langle \sigma_{i,j} \ (i \neq j \in \{1, \dots, n\}) \mid \sigma_{i,j} = \sigma_{j,i} ; \ \sigma_{i,j}^2 = \sigma_{i,j} ;$$
$$\sigma_{i,j}\sigma_{j,k}\sigma_{k,l} = \sigma_{i,j}\sigma_{i,l}\sigma_{k,l} ;$$
$$\sigma_{i,j}\sigma_{i,k}\sigma_{j,k} = \sigma_{i,j}\sigma_{j,k} ; \ \sigma_{i,j}\sigma_{j,l}\sigma_{i,j} = \sigma_{i,j} ;$$
$$\sigma_{i,j}\sigma_{k,l}\sigma_{i,k} = \sigma_{i,j}\sigma_{j,l}\sigma_{i,k} ;$$
$$\sigma_{i,j}\sigma_{k,l} = \sigma_{k,l}\sigma_{i,j} \ (i,j,k,l \text{ pairwise different}) \rangle$$

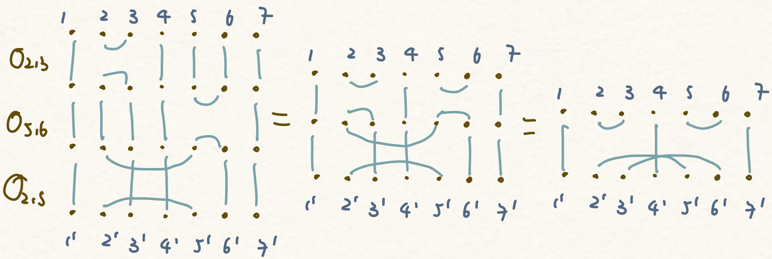
defines $B_n \setminus S_n$.

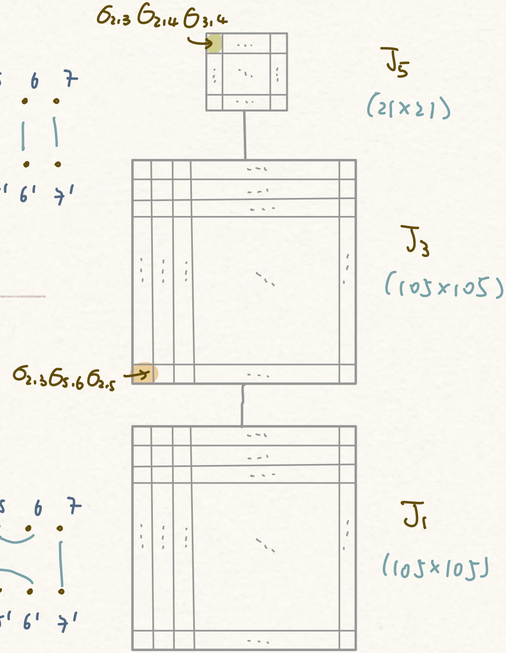
Presentation for $B_7 \setminus S_7 = I_5$

- $\sigma_{2,3}\,\sigma_{3,4} = \sigma_{2,3}\,\sigma_{2,4}\,\sigma_{3,4} \in J_5$



- $\sigma_{2,3}\,\sigma_{5,6}\,\sigma_{2,5} = \sigma_{2,3}\,\sigma_{3,6}\,\sigma_{2,5} \in J_3$





Relational depth of B_n and ideals

Theorem

Let $n \geq 4$. Let $\varepsilon = \begin{cases} 1 & (n \text{ odd}) \\ 0 & (n \text{ even}) \end{cases}$. For $m = \varepsilon, \varepsilon+2, \dots, n$, let I_m be the ideal of all partitions in B_n of rank at most m. Then

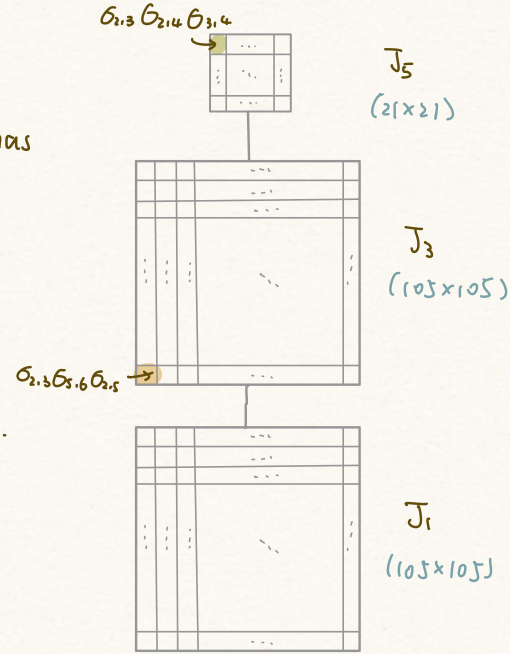
$$\operatorname{depth}(I_m) = \begin{cases} 3 & (m = n) \\ \frac{n-m}{2} + 1 & \left(\frac{n+\varepsilon}{2} < m < n\right) \\ \frac{m-\varepsilon}{2} + 1 & \left(m \leq \frac{n+\varepsilon}{2}\right). \end{cases}$$

Presentation for $B_7 \setminus S_7 = I_5$

The presentation for $B_7 \setminus S_7$ ($= I_5$) given by Maltcev and Mazorchuk has depth $= 2$. In fact,

- If $n = 7$,

  $$\mathrm{depth}(I_5) = \frac{5 - \max(1,3)}{2} + 1 = 2.$$



Example : Relational depth of B_{13} and its ideals

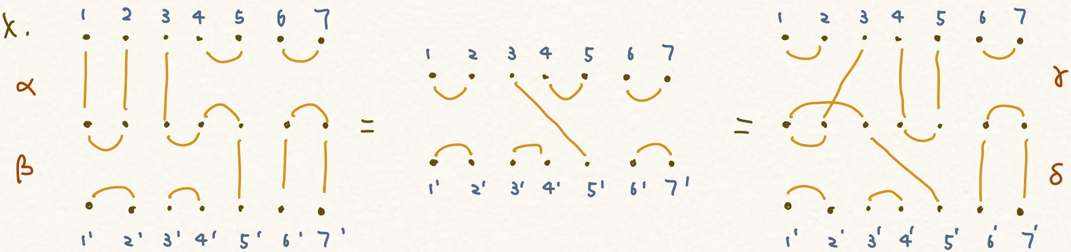
	relational depth	the maximal minimum rank of the defining relations
I_1	1	1
I_3	2	1
I_5	3	1
I_7	4	1
I_9	3	5
I_{11}	2	9
B_{13}	3	9

Idea of the proof

- Find a lower bound.
  - Show that certain 'depth' is not enough
- Find an upper bound.
  - Need to be able to deduce some important relations

    * $x_\alpha x_\beta = x_\gamma x_\delta$

E.x.

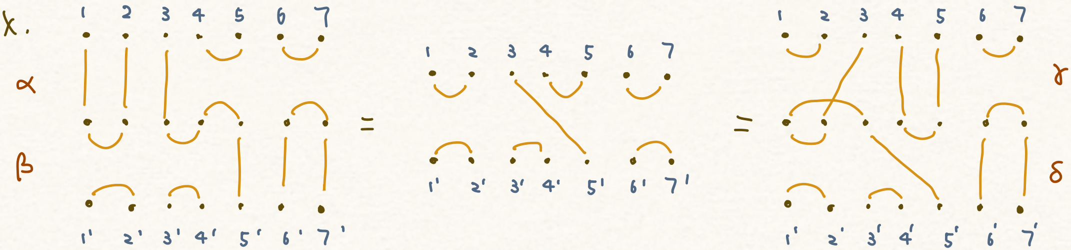


Idea of the proof

- Need to be able to deduce some important relations

* $x_\alpha x_\beta = x_\gamma x_\delta$

E.x.



* $x_\alpha x_\beta = x_{\alpha_1} x_{\alpha_2} x_{\beta_1} x_{\beta_2} \quad (\alpha_1, \alpha_2, \beta_1, \beta_2 \in J_5) = \cdots$

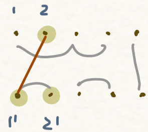
Use relations of partitions of higher ranks

$\cdots = x_{\gamma_1} x_{\gamma_2} x_{\delta_1} x_{\delta_2} = x_\gamma x_\delta$

Other interesting semigroups

- the semigroup of order preserving maps (✓)

- diagram monoids - Partition monoid,



  Partial Brauer monoid. Temperley-Lieb monoid ...

- semigroup of $n \times n$ matrices

⋮

Thank you =)